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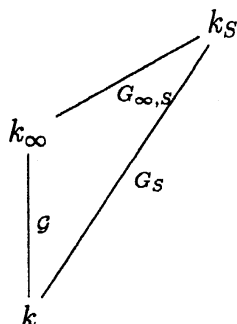
# A spectral sequence for Iwasawa adjoints

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## Abstract

This article is, apart from some small corrections, an unchanged version of a note written in 1994. It was circulated as a manuscript and quoted in several articles. Its aim was to provide a purely algebraic tool for the Iwasawa theory of arbitrary  $p$ -adic Lie groups, by constructing a spectral sequence which relates the generalized 'Iwasawa adjoints'  $E^i(M)$  of certain Iwasawa modules  $M$  to projective limits often encountered in the applications. A somewhat extended version will be published elsewhere.

Let  $k$  be a number field, fix a prime  $p$ , and let  $k_\infty$  be some Galois extension of  $k$  such that  $\mathcal{G} = \text{Gal}(k_\infty/k)$  is a  $p$ -adic Lie-group (e.g.,  $\mathcal{G} \cong \mathbb{Z}_p^r$  for some  $r \geq 1$ ). Let  $S$  be a finite set of primes containing all primes above  $p$  and  $\infty$ , and all primes ramified in  $k_\infty/k$ , and let  $k_S$  be the maximal  $S$ -ramified extension of  $k$ ; by assumption,  $k_\infty \subseteq k_S$ . Let  $G_S = \text{Gal}(k_S/k)$  and  $G_{\infty,S} = \text{Gal}(k_S/k_\infty)$ .



Let  $A$  be a discrete (left)  $G_S$ -module which is isomorphic to  $(\mathbb{Q}_p/\mathbb{Z}_p)^r$  for some  $r \geq 1$  as an abelian group (e.g.,  $A = \mathbb{Q}_p/\mathbb{Z}_p$  with trivial action, or  $A = E[p^\infty]$ , the group of  $p$ -power torsion points of an elliptic curve  $E/k$  with good reduction outside  $S$ ). We are not assuming that  $G_{\infty,S}$  acts trivially.

Let  $\Lambda = \mathbb{Z}_p[[\mathcal{G}]]$  be the completed group ring. For a finitely generated  $\Lambda$ -module  $M$  we put

$$E^i(M) = \text{Ext}_\Lambda^i(M, \Lambda).$$

Hence  $E^0(M) = \text{Hom}_\Lambda(M, \Lambda) =: M^+$  is just the  $\Lambda$ -dual of  $M$ . This has a natural structure of a  $\Lambda$ -module, by letting  $\sigma \in \mathcal{G}$  act via

$$\sigma f(m) = f(m) \cdot \sigma^{-1}$$

for  $f \in M^+$ ,  $m \in M$ . It is known that  $\Lambda$  is a noetherian ring (here we use that  $\mathcal{G}$  is a  $p$ -adic Lie group), by results of Lazard [La]. Hence  $M^+$  is a finitely generated  $\Lambda$ -module again (choose a projection  $\Lambda^r \twoheadrightarrow M$ ; then we have an injection  $M^+ \hookrightarrow (\Lambda^r)^+ = \Lambda^r$ ). By standard homological algebra, the  $E^i(M)$  are finitely generated  $\Lambda$ -modules for all  $i \geq 0$ .

**Examples** a) If  $\mathcal{G} = \mathbb{Z}_p$ , then  $\Lambda = \mathbb{Z}_p[[\mathcal{G}]] \cong \mathbb{Z}_p[[X]]$  is the classical Iwasawa algebra, and, for a  $\Lambda$ -torsion module  $M$ ,  $E^1(M)$  is isomorphic to the Iwasawa adjoint, which can be defined as

$$\text{ad}(M) = \varprojlim_n (M/\alpha_n M)^\vee$$

where  $(\alpha_n)_{n \in \mathbb{N}}$  is a sequence of elements in  $\Lambda$  such that  $\lim_{n \rightarrow \infty} \alpha_n = 0$  and  $(\alpha_n)$  is prime to the support of  $M$  for every  $n \geq 1$ , and where

$$N^\vee = \text{Hom}(N, \mathbb{Q}_p/\mathbb{Z}_p)$$

is the Pontrjagin dual of a compact  $\mathbb{Z}_p$ -module  $N$ . For any finitely generated  $\Lambda$ -module  $M$ ,  $E^1(M)$  is quasi-isomorphic to  $\text{Tor}_\Lambda(M)^\sim$ , where  $\text{Tor}_\Lambda(M)$  is the  $\Lambda$ -torsion submodule of  $M$ , and  $M^\sim$  is the "Iwasawa twist" of a  $\Lambda$ -module  $M$ : the action of  $\gamma \in \mathcal{G}$  is changed to the action of  $\gamma^{-1}$ .

b) If  $\mathcal{G} = \mathbb{Z}_p^r$ ,  $r \geq 1$ , then the  $E^i(M)$  are the standard groups considered in local duality. By duality for the ring  $\mathbb{Z}_p[[\mathcal{G}]] = \mathbb{Z}_p[[x_1, \dots, x_r]]$ , they can be computed in terms of local cohomology groups (with support) or by a suitable Koszul complex.

The topic of this note is the following observation.

**Theorem 1** *There is a spectral sequence*

$$E_2^{p,q} = E^p(H^q(G_{\infty,S}, A)^\vee) \Rightarrow \varprojlim_{k',m} H^{p+q}(G_S(k'), A[p^m]).$$

Here the limit runs through the natural numbers  $m$  and the finite extensions  $k'/k$  contained in  $k_\infty$ , via the natural maps

$$H^n(G_S(k'), A[p^{m+1}]) \rightarrow H^n(G_S(k'), A[p^m]) \quad (m \geq 1)$$

and the corestrictions.

Before we give the proof, we discuss what this spectral sequence gives in more down-to-earth terms. First of all, we always have the 5-low-terms exact sequence

$$\begin{aligned} 0 &\rightarrow E^1(H^0(G_{\infty,S}, A)^\vee) \xrightarrow{\text{inf}^1} \varprojlim_{k'} H^1(G_S(k'), T_p A) \\ &\rightarrow (H^1(G_{\infty,S}, A)^\vee)^+ \rightarrow E^2(H^0(G_{\infty,S}, A)^\vee) \xrightarrow{\text{inf}^2} \varprojlim_{k'} H^2(G_S(k'), T_p A). \end{aligned}$$

To say more, we consider some assumptions.

**A.1** Assume that  $p > 2$  or that  $k_\infty$  is totally imaginary. Then

$$H^r(G_{\infty,S}, A) = 0 = \varprojlim_{k'} H^r(G_S(k'), T_p A)$$

for all  $r > 2$ .

**Corollary 2** *Assume that  $H^2(G_{\infty,S}, A) = 0$  (This is the so-called "weak Leopoldt conjecture" for  $A$ . It is stated classically for  $A = \mathbb{Q}_p/\mathbb{Z}_p$ , and there are precise conjectures when this is expected for representations coming from algebraic geometry, cf. [Ja2]).*

Then the cokernel of  $\inf^2$  is

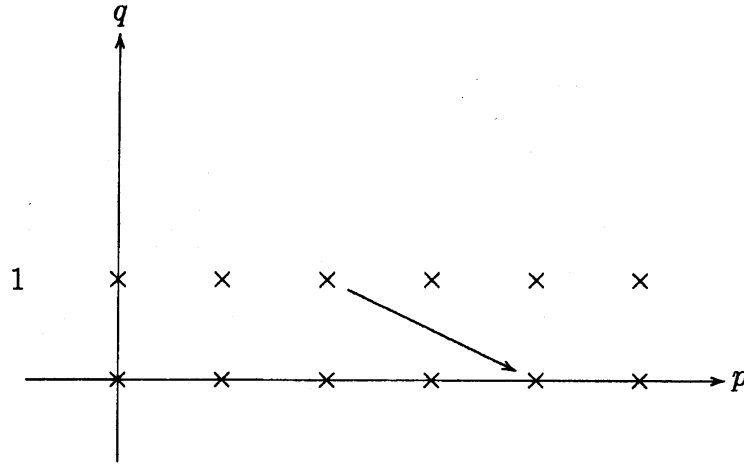
$$\ker(E^1(H^1(G_{\infty, S}, A)^\vee) \rightarrow E^3(H^0(G_{\infty, S}, A)^\vee)),$$

and there are isomorphisms

$$E^i(H^1(G_{\infty, S}, A)^\vee) \xrightarrow{\sim} E^{i+2}(H^0(G_{\infty, S}, A)^\vee)$$

for  $i \geq 2$ .

**Proof** This comes from A.1 and the following picture of the spectral sequence



**Corollary 3** Assume that  $H^0(G_{\infty, S}, A) = 0$ . Then

(a)

$$\varprojlim_{k'} H^1(G_S(k'), T_p A) \xrightarrow{\sim} H^1(G_{\infty, S}, A)^\vee{}^+$$

(b) There is an exact sequence

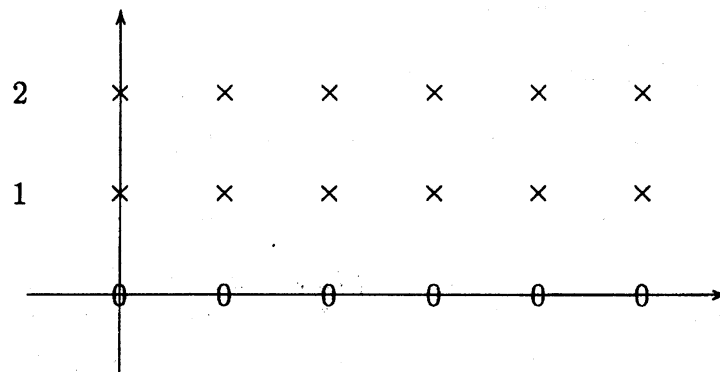
$$\begin{aligned} 0 \rightarrow E^1(H^1(G_{\infty, S}, A)^\vee) &\rightarrow \varprojlim_{k'} H^2(G_S(k'), T_p A) \\ &\rightarrow (H^2(G_{\infty, S}, A)^\vee)^+ \rightarrow E^2(H^1(G_{\infty, S}, A)^\vee) \rightarrow 0 \end{aligned}$$

(c) There are isomorphisms

$$E^i(H^2(G_{\infty, S}, A)^\vee) \xrightarrow{\sim} E^{i+2}(H^1(G_{\infty, S}, A)^\vee)$$

for  $i \geq 1$ .

**Proof** In this case, the spectral sequence looks like



**Corollary 4** Assume that  $\mathcal{G}$  is a  $p$ -adic Lie group of dimension 1 (equivalently: an open subgroup is  $\cong \mathbb{Z}_p$ ). Then  $E^i(-) = 0$  for  $i \geq 3$ . Let

$$B = \text{im} (\inf^2 : E^2(H^0(G_{\infty, S}, A)^\vee) \rightarrow \varprojlim_{k'} H^2(G_S(k'), T_p A))$$

Then  $B$  is finite, and there is an exact sequence

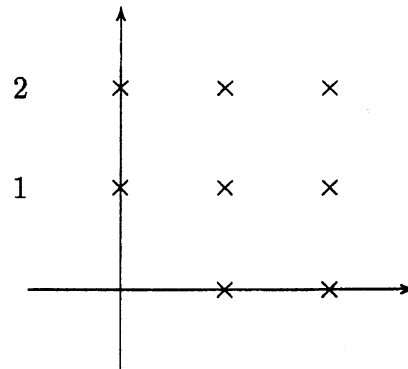
$$\begin{aligned} 0 \rightarrow E^1(H^1(G_{\infty, S}, A)^\vee) &\rightarrow \varprojlim_{k'} H^2(G_S(k'), T_p A)/B \rightarrow (H^2(G_{\infty, S}, A)^\vee)^+ \\ &\rightarrow E^2(H^1(G_{\infty, S}, A)^\vee) \rightarrow 0, \end{aligned}$$

and

$$E^1(H^2(G_{\infty, S}, A)^\vee) = 0 = E^2(H^2(G_{\infty, S}, A)^\vee),$$

i.e.,  $(H^2(G_{\infty, S}, A)^\vee)$  is a projective  $\Lambda$ -module.

**Proof** Quite generally, for a  $p$ -adic Lie group  $\mathcal{G}$  of dimension  $n$  one has  $\text{vcd}_p(\mathcal{G}) = n$  for the virtual cohomological  $p$ -dimension of  $\mathcal{G}$ , and hence  $E^i(-) = 0$  for  $i > n + 1$ , cf. [Ja3]. The finiteness of  $E^2(M)$  (for a finitely generated  $\Lambda$ -module  $M$ ) in our case is well-known, cf. [Ja3]. The remaining claims follow from the following shape of the spectral sequence:



**Lemma 5** Assume that  $\mathcal{G}$  is a  $p$ -adic Lie group of dimension  $n$  (e.g.,  $\mathcal{G}$  contains an open subgroup  $\cong \mathbb{Z}_p^n$ ). Then  $E^i(H^0(G_{\infty, S}, A)^\vee) = 0$  for  $i \neq n, n + 1$ .

(a) If  $H^0(G_{\infty, S}, A)$  is divisible (e.g., if  $G_{\infty, S}$  acts trivially on  $A$ ), then

$$E^i(H^0(G_{\infty, S}, A)^\vee) = \begin{cases} 0 & \text{for } i \neq n \\ \text{Hom}(D, H^0(G_{\infty, S}, A)), & \text{for } i = n, \end{cases}$$

where  $D$  is the dualising module for  $\mathcal{G}$  ( $D = \mathbb{Q}_p/\mathbb{Z}_p$  if  $\mathcal{G} = \mathbb{Z}_p^n$ ).

(b) If  $H^0(G_{\infty, S}, A)$  is finite, then

$$E^i(H^0(G_{\infty, S}, A)^\vee) = \begin{cases} 0 & \text{for } i \neq n + 1 \\ \text{Hom}(H^0(G_{\infty, S}, A), D)^\vee & \text{for } i = n + 1 \end{cases}$$

**Proof** This is well-known, see [Ja3].

**Corollary 6** Let  $\mathcal{G}$  is a  $p$ -adic Lie group of dimension 2 (e.g.,  $\mathcal{G}$  contains an open subgroup  $\cong \mathbb{Z}_p^2$ ). If  $G_{\infty, S}$  acts trivially on  $A$ , then there are exact sequences

$$\begin{aligned} 0 \rightarrow \varprojlim_{k'} H^1(G_S(k'), T_p A) &\rightarrow (H^1(G_{\infty, S}, A)^\vee)^+ \\ &\rightarrow T_p A \xrightarrow{\inf^2} \varprojlim_{k'} H^2(G_S(k'), T_p A) \end{aligned}$$

and

$$\begin{aligned} 0 \rightarrow E^1(H^1(G_{\infty, S}, A)^\vee) &\rightarrow \varprojlim_{k'} H^2(G_S(k'), T_p A) / \text{im } \inf^2 \\ &\rightarrow (H^1(G_{\infty, S}, A)^\vee)^+ \rightarrow E^2(H^1(G_{\infty, S}, A)^\vee) \rightarrow 0, \end{aligned}$$

an isomorphism

$$E^1(H^2(G_{\infty, S}, A)^\vee) \xrightarrow{\sim} E^3(H^1(G_{\infty, S}, A)^\vee),$$

and one has

$$E^2(H^2(G_{\infty, S}, A)^\vee) = 0 = E^3(H^2(G_{\infty, S}, A)^\vee).$$

**Proof** The spectral sequence looks like

**Corollary 7** Let  $\mathcal{G}$  be a  $p$ -adic Lie group of dimension 2 (So  $E^i(-) = 0$  for  $i \geq 4$ ). If  $H^0(G_{\infty, S}, A)$  is finite, then

$$\varprojlim_{k'} H^1(G_S(k'), T_p A) \cong (H^1(G_{\infty, S}, A)^\vee)^+.$$

If

$$d_2^{1,1} : E^1(H^1(G_{\infty, S}, A)^\vee) \rightarrow E^3(H^0(G_{\infty, S}, A)^\vee)$$

is the differential of the spectral sequence in the theorem, then one has an exact sequence

$$\begin{aligned} 0 \rightarrow \ker d_2^{1,1} &\rightarrow \varprojlim_{k'} H^2(G_S(k'), T_p A) \rightarrow \\ &\rightarrow \ker(d_2^{0,2} : (H^2(G_{\infty, S}, A)^\vee)^+ \rightarrow E^2(H^1(G_{\infty, S}, A)^\vee)) \rightarrow \text{coker } d_2^{1,1} \rightarrow 0, \end{aligned}$$

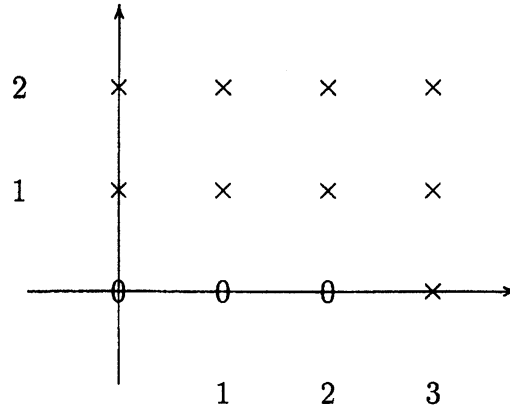
an isomorphism

$$E^1(H^2(G_{\infty, S}, A)^\vee) \xrightarrow{\sim} E^3(H^1(G_{\infty, S}, A)^\vee),$$

and the vanishing

$$E^2(H^2(G_{\infty, S}, A)^\vee) = 0 = E^3(H^2(G_{\infty, S}, A)^\vee).$$

**Proof** The spectral sequence looks like



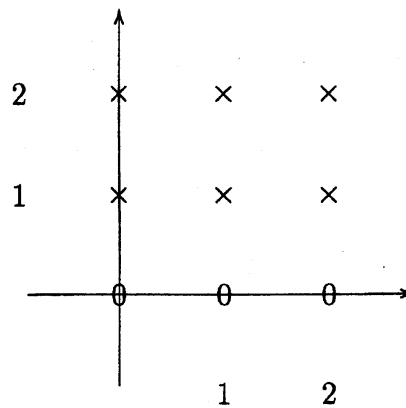
**Remark** In the situation of Corollary 5, one has an exact sequence up to *finite modules*:

$$\begin{aligned} 0 \rightarrow E^1(H^1(G_{\infty, S}, A)^\vee) &\rightarrow \varprojlim_{k'} H^2(G_S(k'), T_p A) \\ &\rightarrow (H^2(G_{\infty, S}, A)^\vee)^+ \rightarrow E^2(H^1(G_{\infty, S}, A)^\vee) \rightarrow 0. \end{aligned}$$

**Corollary 8** Let  $\mathcal{G}$  be a  $p$ -adic Lie group of dimension  $> 2$ . Then

$$(H^1(G_{\infty, S}, A)^\vee)^+ \cong \varprojlim_{k'} H^1(G_S(k'), T_p A)$$

**Proof** The first three columns of the spectral sequence look like



We now turn to the **proof of Theorem 1**:

Let  $G$  be any profinite group, let  $H \leq G$  be a closed normal subgroup, and denote  $\mathcal{G} = G/H$ . Let  $M_G$  be the category of discrete  $G$ -modules which are torsion as abelian groups, and let  $M_G^{\mathbb{N}}$  be the category of inverse systems  $(A_n)$  in  $M_G$  as in [Ja1]. For any  $A$  in  $M_G$ , one gets an inverse system  $T_p A := (A[p^n])$ , where the transition maps  $A[p^{n+1}] \rightarrow A[p^n]$  are induced by multiplication with  $p$  in  $A$ . For reasons explained later, denote by  $H_{\text{cont}}^m(G, H; RT_p A)$  the value at  $A$  of the  $m$ -th derived functor of the left exact functor

$$F : A \mapsto \varprojlim_n \varprojlim_U H^0(U, A[p^n])$$

where  $U$  runs through all open subgroups  $U \subset G$  containing  $H$ , and the transition maps are the corestriction maps and these coming from  $A[p^{n+1}] \rightarrow A[p^n]$ , respectively. We can write  $F$  as the composition of the functors

$$\begin{aligned} T_p : M_G &\rightarrow M_G^{\mathbb{N}} \\ A &\mapsto (A[p^n]) \end{aligned}$$

and

$$\begin{aligned} H_{\text{cont}}^0(G, H; -) : M_G^{\mathbb{N}} &\rightarrow \text{Ab} \\ (A_n) &\mapsto \varprojlim_n \varprojlim_U H^0(U, A_n) \end{aligned}$$

where  $\text{Ab}$  is the category of abelian groups. Because  $T_p$  maps injectives to  $H_{\text{cont}}^0(G, H; -)$ -acyclics we get a spectral sequence

$$E_2^{p,q} = H_{\text{cont}}^p(G, H; R^q T_p A) \Rightarrow H_{\text{cont}}^{p+q}(G, H; RT_p A).$$

One has

$$R^q T_p A = \begin{cases} (A/p^m A) & q = 1 \\ 0 & q > 1 \end{cases}$$

and hence short exact sequences

$$0 \rightarrow H_{\text{cont}}^n(G, H; T_p A) \rightarrow H_{\text{cont}}^n(G, H; RT_p A) \rightarrow H_{\text{cont}}^{n-1}(G, H; R^1 T_p A) \rightarrow 0$$

explaining the notation for  $R^n F$ . In fact,  $H_{\text{cont}}^n(G, H; RT_p A)$  is the hypercohomology with respect to  $H_{\text{cont}}^0(G, H; -)$  of a complex  $RT_p A$  in  $M_G^{\mathbb{N}}$  computing the  $R^i T_p A$ . If  $A$  is  $p$ -divisible, then  $R^q T_p A = 0$  for all  $q > 0$ , the spectral sequence degenerates and gives isomorphisms  $H_{\text{cont}}^n(G, H; RT_p A) \cong H_{\text{cont}}^n(G, H; T_p A)$ . There is another spectral sequence from deriving the inverse limit, viz.

$$E_2^{p,q} = R^p \varprojlim_{n,U} H^q(U, A_n) \Rightarrow H_{\text{cont}}^{p+q}(G, H; (A_n))$$

If  $G/H$  has a countable basis of neighbourhoods of identity, i.e., if there is a countable family  $U_\nu$  of open subgroup,  $H \leq U_\nu \leq G$ , with  $\bigcap_\nu U_\nu = H$ , then  $R^p \varprojlim_{n,U} = 0$  for  $p > 1$ , and  $R^1 \varprojlim_{n,U}$  has the usual description ([Ja1]). If, in addition, all  $H^q(U, A_n)$  are finite, then  $R^1 \varprojlim_{n,U} H^q(U, A_n) = 0$ , and we get isomorphisms

$$H_{\text{cont}}^n(G, H; (A_n)) = \varprojlim_n \varprojlim_U H^n(U, A_n).$$



All this applies in the situation of the theorem, so that

$$H_{\text{cont}}^n(G_S, G_{\infty, S}; RT_p A) = \varprojlim_n \varprojlim_U H^n(G_S(k'), A[p^m])$$

where  $k'$  runs through the finite extensions  $k'/k$  contained in  $k_{\infty}$ .

On the other hand, we can write  $F$  as the composition of the left exact functors

$$\begin{array}{ccc} H^0(H, -) : M_G & \rightarrow & M_G \\ A & \mapsto & A^H \end{array}$$

and

$$\begin{array}{ccc} M_G & \rightarrow & Ab \\ B & \mapsto & \text{Hom}_{\Lambda}(B^{\vee}, \Lambda) \end{array}$$

where  $\Lambda = \mathbb{Z}_p[[\mathcal{G}]]$  is the completed  $\mathbb{Z}_p$ -group ring of  $\mathcal{G}$  and  $B^{\vee}$  is the Pontrjagin dual of  $B$ , which is a compact  $\Lambda$ -module. In fact, one has (cf. [Ja3] p. 179).

$$\begin{aligned} \text{Hom}_{\Lambda}(B^{\vee}, \Lambda) &= \varprojlim_U \text{Hom}_{\mathcal{G}}(B^{\vee}, \mathbb{Z}_p[G/U]) \\ &= \varprojlim_n \varprojlim_U \text{Hom}(H^0(U/H, B)^{\vee}, \mathbb{Z}/p^n \mathbb{Z}) \\ &= \varprojlim_n \varprojlim_U \text{Hom}(H^0(U/H, B[p^n])^{\vee}, \mathbb{Z}/p^n \mathbb{Z}) \\ &= \varprojlim_n \varprojlim_U H^0(U/H, B[p^n]) \end{aligned}$$

and hence

$$\text{Hom}_{\Lambda}(H^0(H, A)^{\vee}, \Lambda) = \varprojlim_n \varprojlim_U (H^0(U, A[p^n]) = F(A).$$

Since taking Pontrjagin duals is an exact functor

$$M_G \rightarrow (\text{compact } \Lambda\text{-modules})$$

taking injectives to projectives, the derived functors of the functor  $B \mapsto \text{Hom}_{\Lambda}(B^{\vee}, \Lambda)$  are the functors  $B \mapsto \text{Ext}_{\Lambda}^i(B^{\vee}, \Lambda) = E^i(B^{\vee})$ , and we get a spectral sequence

$$E_2^{p,q} = E^p(H^q(H, A)^{\vee}) \Rightarrow R^{p+q}F(A) = H_{\text{cont}}^{p+q}(G, H; RT_p A).$$

The theorem follows.

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